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On Generalized Linearity of Quadratic Fractional Functions

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Abstract. Quadratic fractional functions are proved to be quasilinear if and only if they are pseudolinear. For these classes of functions, some characterizations are provided by means of the inertia of the quadratic form and the behavior of the gradient of the function itself. The study is then developed showing that generalized linear quadratic fractional functions share a particular structure. Therefore it is possible to suggest a sort of "canonical form" for those functions. A wider class of functions given by the sum of a quadratic fractional function and a linear one is also studied. In this case generalized linearity is characterized by means of simple conditions. Finally, it is deepened on the role played by generalized linear quadratic fractional functions in optimization problems.

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1. Introduction

Quadratic fractional programming deals with constrained optimization problems where the objective function is the ratio of a quadratic function and an affine one. Due to its importance in application models (such as risk theory, portfolio theory, location models), this particular class of nonlinear programs has been widely studied from both a theoretical and an algorithmic point of view (see for example [2, 3, 23]).

Many solution methods have been given for quadratic fractional programming problems whose feasible region is a polyhedron. In these cases the generalized convexity of the objective function plays a fundamental role, since it guarantees the global optimality of local optima.

Among generalized convex functions, the generalized linear ones are extremely useful since the above nice property holds for both maximum and minimum problems. We recall that a function $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is an open convex set, is said to be *pseudolinear* if it is both pseudoconcave and pseudoconvex while it is said to be *quasilinear* if it is both quasiconcave and quasiconvex.

It is known (see for example [1]) that when f is a differentiable pseudolinear function it results that

- f is not a constant function if and only if $\nabla f(x) \neq 0$ for every $x \in A$.

On the other hand, given a closed set $X \subseteq A$, if f is a pseudolinear but not a constant function then the following properties hold:

- there are neither local maxima nor local minima in the interior of X,
- if X is a polyhedral set then the maximum and minimum values are reached on a vertex.

Thanks to their properties, generalized linear functions play a key role in both finding optimality conditions and implementing algorithms for applications.

In this paper we aim to characterize the generalized linearity of quadratic fractional functions and to establish necessary and sufficient conditions which can be easily checked. By means of the proposed characterizations we first prove that a quadratic fractional function is quasilinear if and only if it is pseudolinear. Secondly we show that a quadratic fractional function is pseudolinear if and only if it can be rewritten as the sum of a linear function and a linear fractional one with constant numerator. This result allows us to give conditions characterizing the generalized linearity of a larger class of functions given by the sum of a quadratic fractional function and a linear one.

Furthermore we show that both minimization and maximization problems, involving a generalized linear quadratic fractional function, can be simply solved through equivalent linear ones.

2. Preliminary results

In the next section some new characterizations of generalized linear quadratic fractional functions will be given using the inertia of symmetric matrices. With this regards, the number of the negative eigenvalues of a symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is denoted by $\nu_-(Q)$. Similarly $\nu_+(Q)$ represents the number of the positive eigenvalues, while $\nu_0(Q)$ is the algebraic multiplicity of the 0 eigenvalue. To avoid trivial cases we assume $n \geqslant 2$. A key tool in our study is the following result given by Crouzeix (see [14], Theorem 7, p. 253).

PROPOSITION 1. *Let* $h \in \mathbb{R}^n$, $h \neq 0$, *let* $Q \in \mathbb{R}^{n \times n}$ *be a symmetric matrix and denote with* $Q^\sharp$ *the Moore–Penrose pseudoinverse of* Q.[1] *Then the following*

[1] Let $Q \in \mathbb{R}^{m \times n}$. The Moore–Penrose pseudoinverse of Q is the unique matrix $Q^\sharp \in \mathbb{R}^{n \times m}$ verifying the following Moore–Penrose equations [20–22]:

$$QQ^\sharp Q = Q, \quad Q^\sharp Q Q^\sharp = Q^\sharp, \quad QQ^\sharp = (QQ^\sharp)^T, \quad Q^\sharp Q = (Q^\sharp Q)^T.$$

implication

$$h^T v = 0 \;\Rightarrow\; v^T Q v \geqslant 0$$

is satisfied $\forall v \in \mathbb{R}^n$ *if and only if one of the following conditions holds*:[2]
(i) $\nu_-(Q) = 0$,
(ii) $\nu_-(Q) = 1$, $h \in Q(\mathbb{R}^n)$ *and* $h^T Q^\sharp h \leqslant 0$.

The assumption $h \neq 0$ in Proposition 1 leads to some technical difficulties in the application of the proposition to particular problems where the vector h is not necessarily different from zero. For this reason we state the following corollary which improves the result by Crouzeix [14] not requiring the vector h to be different from zero.

COROLLARY 2. *Let* $h \in \mathbb{R}^n$ *and let* $Q \in \mathbb{R}^{n \times n}$ *be a symmetric matrix. Then the following implication*

$$h^T v = 0 \;\Rightarrow\; v^T Q v \geqslant 0 \tag{1}$$

is satisfied $\forall v \in \mathbb{R}^n$ *if and only if one of the following conditions holds*:
(i) $\nu_-(Q) = 0$,
(ii) $\nu_-(Q) = 1$, $h \neq 0$, $h \in Q(\mathbb{R}^n)$ *and* $u^T Q u \leqslant 0$ $\forall u \in \mathbb{R}^n$ *s.t.* $Qu = h$.

Proof. $\Leftarrow$) If $\nu_-(Q) = 0$ then (1) is trivially satisfied; if (ii) holds then the results follows from Proposition 1 since condition

$$u^T Q u \leqslant 0 \;\forall u \in \mathbb{R}^n \text{ such that } Qu = h$$

implies $h^T Q^\sharp h \leqslant 0$.

$\Rightarrow$) If $\nu_-(Q) = 0$ then (i) holds. Let us now assume $\nu_-(Q) \neq 0$ and suppose by contradiction $h = 0$; then $h^T v = 0$ $\forall v \in \mathbb{R}^n$. Hence for condition (1) $v^T Q v \geqslant 0$ $\forall v \in \mathbb{R}^n$, that is to say that Q is positive semidefinite which is a contradiction being $\nu_-(Q) \neq 0$. Therefore it yields that $h \neq 0$ and (ii) follows from (ii) of Proposition 1. □

Corollary 2 allows us to state the following property which is a key tool in characterizing the generalized linearity of quadratic fractional functions.

COROLLARY 3. *Let* $h \in \mathbb{R}^n$ *and let* $Q \in \mathbb{R}^{n \times n}$, $Q \neq 0$, *be a symmetric matrix. The following implication*

$$h^T v = 0 \;\Rightarrow\; v^T Q v = 0 \tag{2}$$

is satisfied $\forall v \in \mathbb{R}^n$ *if and only if one of the following conditions holds*:
(i) $\nu_0(Q) = n - 1$, $h \neq 0$ *and* $h \in Q(\mathbb{R}^n)$,

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In [4, 5] matrix $Q^\sharp$ has been called Moore–Penrose generalized inverse of Q.

[2] Recall that $Q(\mathbb{R}^n) = \{y \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \text{ such that } y = Qx\} = \mathrm{Im}(Q)$.

(ii) $\nu_-(Q)=\nu_+(Q)=1$, $h\neq 0$, $h\in Q(\mathbb{R}^n)$ *and* $u^TQu=0$ $\forall u\in\mathbb{R}^n$ *such that* $Qu=h$.

Proof. First note that, from Corollary 2, the implication

$$h^Tv=0 \quad\Rightarrow\quad v^TQv\leqslant 0$$

is satisfied $\forall v\in\mathbb{R}^n$ if and only if one of the following conditions holds:

(a) $\nu_+(Q)=0$,

(b) $\nu_+(Q)=1$, $h\neq 0$, $h\in Q(\mathbb{R}^n)$ and $u^TQu\geqslant 0$ $\forall u\in\mathbb{R}^n$ s.t. $Qu=h$.

$\Leftarrow$) If (i) holds and Q is positive semidefinite then $\nu_+(Q)=1$, $\nu_-(Q)=0$, $h\neq 0$, $h\in Q(\mathbb{R}^n)$ and $u^TQu\geqslant 0$ $\forall u\in\mathbb{R}^n$. Thus (i) of Corollary 2 and condition *(b)* hold, hence $h^Tv=0$ implies $v^TQv\geqslant 0$ and $v^TQv\leqslant 0$, so that (2) holds. The case Q negative semidefinite can be proved with the same arguments. If (ii) holds then both conditions *(b)* and (ii) of Corollary 2 are satisfied; again $h^Tv=0$ implies $v^TQv\geqslant 0$ and $v^TQv\leqslant 0$ so that (2) holds.

$\Rightarrow$*)* First note that Condition (2) holds if and only if

$$\{h^Tv=0 \quad\Rightarrow\quad v^TQv\geqslant 0\} \quad\text{and}\quad \{h^Tv=0 \quad\Rightarrow\quad v^TQv\leqslant 0\}$$

and this happens if and only if one of conditions (i) and (ii) of Corollary 2 holds together with one of conditions *(a)* and *(b)*. Observe that conditions *(a)* and (i) of Corollary 2 imply $Q=0$ which is a contradiction. Conditions *(a)* and (ii) of Corollary 2 imply condition (i) and the same happens if *(b)* and (i) of Corollary 2 hold. If otherwise conditions *(b)* and (ii) of Corollary 2 are satisfied then condition (ii) follows immediately. Since all the possible exhaustive cases have been considered, the result is proved. □

Other useful results, which are worth recalling, are the following ones by Diewert et al. [15] (Corollary 4.3 p. 401, and Theorem 10 p. 407).

PROPOSITION 4. *Let f be a differentiable function defined on the open convex set $A\subset\mathbb{R}^n$. Then:*

(i) *f is quasiconvex if and only if $\forall x\in A$, $\forall v\in\mathbb{R}^n\setminus\{0\}$, such that $\nabla f(x)^Tv=0$ the function $\phi_v(t)=f(x+tv)$ does not attain a semistrict local maximum at $t=0$;*[3]

Suppose function f to be also continuously differentiable, then:

(ii) *f is pseudoconvex if and only if $\forall x\in A$, $\forall v\in\mathbb{R}^n\setminus\{0\}$, such that $\nabla f(x)^Tv=0$ the function $\phi_v(t)=f(x+tv)$ attains a local minimum at $t=0$.*

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[3] Let f be defined on the open interval $(a,b)\subset\mathbb{R}$. Then f is said to attain a *semistrict local maximum* at a point $x_0\in(a,b)$ if there exists two points $x_1,x_2\in(a,b)$, $x_1<x_0<x_2$, such that

$$f(x_0)\geqslant f(x_2+\lambda(x_1-x_2)) \quad \forall\lambda\in[0,1]$$

and $f(x_0)>\min\{f(x_1),f(x_2)\}$.

Recall finally the following very well known characterization of pseudolinear functions (see for example [1, 19]) which follows from the results by Diewert et al. [15].

PROPOSITION 5. *A differentiable function* $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ *an open convex set, is pseudolinear if and only if the following implication holds* $\forall x \in A$, $\forall v \in \mathbb{R}^n$, $v \neq 0$, $\forall t \in \mathbb{R}$ *such that* $x + tv \in A$*:*

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad \phi_v(t) = f(x + tv) \text{ is constant}$$

3. Generalized Linearity of Quadratic Fractional Functions

In this section we are going to characterize the generalized linearity of quadratic fractional functions of the following kind:[4]

$$f(x) = \frac{\frac{1}{2}x^T Qx + q^T x + q_0}{b^T x + b_0} \tag{3}$$

defined on the set $X = \{x \in \mathbb{R}^n : b^T x + b_0 > 0\}$, $n \geqslant 2$, where $Q \neq 0$ is a $n \times n$ symmetric matrix, $q, x, b \in \mathbb{R}^n$, $b \neq 0$, and $q_0, b_0 \in \mathbb{R}$. Note that being Q symmetric, it is $Q \neq 0$ if and only if $\nu_0(Q) \leqslant n - 1$. Moreover it results:

$$\nabla f(x) = \frac{Qx + q - f(x)b}{b^T x + b_0} \tag{4}$$

REMARK 6. It is important to specify that f in (3) is not a constant function. Suppose by contradiction that f is a constant, that is $f(x) = k$, so that $\nabla f(x) = 0$ $\forall x \in X$. Consider now an arbitrary $x_1 \in X$ and let $\alpha \in \mathbb{R}$ be such that $\alpha \neq 0, \alpha \neq 1$ and $\alpha x_1 \in X$. From (4) it results $Qx_1 + q - kb = Q\alpha x_1 + q - kb$ and hence $Qx_1 = \alpha Qx_1$ which implies $Qx_1 = 0$, i.e. $Qx = 0$ $\forall x \in X$. Since X is an n-dimensional halfspace it must be $Q = 0$ which contradicts the definition of (3).

The next theorem points out that a quadratic fractional function is pseudolinear if and only if it is quasilinear. Moreover, for these classes of functions we give a new characterization based on the beavior of $v^T Qv$ when $\nabla f(x)^T v = 0$.

THEOREM 7. *Consider function* f *defined in* (3). *Then the following conditions are equivalent*:

(i) f *is pseudolinear on* X,

(ii) f *is quasilinear on* X,

(iii) *the following implication holds* $\forall x \in X$ $\forall v \in \mathbb{R}^n \setminus \{0\}$*:*

$$\nabla f(x)^T v = 0 \quad \Rightarrow \quad v^T Qv = 0,$$

[4] Pseudoconvexity of this class of function has been recently studied in [7] and [11].

(iv) *one of the following conditions holds* $\forall x \in X$:

(*a*) $\nu_0(Q) = n-1$, $\nabla f(x) \neq 0$ *and* $\nabla f(x) \in Q(\mathbb{R}^n)$,

(*b*) $\nu_-(Q) = \nu_+(Q) = 1$, $\nabla f(x) \neq 0$, $\nabla f(x) \in Q(\mathbb{R}^n)$ *and* $u^T Q u = 0$ $\forall u \in \mathbb{R}^n$ *such that* $Qu = \nabla f(x)$.

Proof. First note that, from (4), it results:

$$\phi_v(t) = f(x+tv) = f(x) + \frac{\frac{1}{2}t^2 v^T Q v + t(b^T x + b_0)\nabla f(x)^T v}{b^T x + b_0 + t b^T v}.$$

(i)⇒(ii) Trivial.

(ii)⇒(iii) Since f is both quasiconvex and quasiconcave, it follows from Proposition 4 that the function $\phi_v(t) = f(x) + \frac{1}{2}\frac{t^2 v^T Q v}{b^T x + b_0 + t b^T v}$ does not attain either a semistrict local maximum or a semistrict local minimum at $t=0$, $\forall x \in X$, $\forall v \in \mathbb{R}^n \setminus \{0\}$ such that $\nabla f(x)^T v = 0$. This happens only if $v^T Q v = 0$, being $b^T x + b_0 + t b^T v > 0$ $\forall t \in \mathbb{R}$ such that $x + tv \in X$.

(iii)⇒(i) Since $\nabla f(x)^T v = 0$ implies $v^T Q v = 0$, it follows that $\phi_v(t) = f(x)$ $\forall t \in \mathbb{R}$ such that $x + tv \in X$. Hence the results follows from Proposition 5.

(iii)⇔(iv) Follows directly from Corollary 3. □

Recall that condition $\nabla f(x) \neq 0$ $\forall x \in X$, is a necessary condition for a not constant pseudolinear function.

Now we aim to prove that all the generalized linear quadratic fractional functions can be rewritten in the same way. In other words we are going to point out the existence of a sort of canonical form for these functions. We first prove the following lemma.

LEMMA 8. *Let us consider function* f *defined in (3). Then:*

(i) $\nabla f(x) \in Q(\mathbb{R}^n)$ $\forall x \in X$ *if and only if* $\exists \overline{x}, \overline{y} \in \mathbb{R}^n$ *such that* $Q\overline{x} = q$ *and* $Q\overline{y} = b$.

In particular, for any given $x \in X$:

(ii) $Qu = \nabla f(x)$ *if and only if* $u = \frac{x + \overline{x} - f(x)\overline{y}}{b^T x + b_0} + k$ *with* $k \in \ker(Q)$,

(iii) $b^T x = b^T \overline{x}$ *and* $q^T x = q^T \overline{x}$ *for all* $x \in \mathbb{R}^n$ *such that* $Qx = q$,

(iv) $b^T y = b^T \overline{y}$ *and* $q^T y = q^T \overline{y}$ *for all* $y \in \mathbb{R}^n$ *such that* $Qy = b$,

(v) $u^T Q u = \frac{p(x)}{\left(b^T x + b_0\right)^2}$ *with:*

$$p(x) = (f(x))^2 b^T \overline{y} + 2f(x)\left[b_0 - b^T \overline{x}\right] + \left(q^T \overline{x} - 2q_0\right) \tag{5}$$

Proof. (i) Assume

$$\nabla f(x) = \frac{Qx + q - f(x)b}{b^T x + b_0} \in Q(\mathbb{R}^n) \ \forall x \in X$$

and let us prove that

$$\exists \overline{x},\ \overline{y} \in \mathbb{R}^n \text{ such that } Q\overline{x} = q \text{ and } Q\overline{y} = b.$$

Since f is not constant (see Remark 6), $\exists x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$. Therefore $\exists u_1, u_2 \in \mathbb{R}^n$ such that

$$Qu_1 = Qx_1 + q - f(x_1)b \quad \text{and} \quad Qu_2 = Qx_2 + q - f(x_2)b.$$

This implies that

$$Q\left(\frac{u_1 - u_2 - x_1 + x_2}{f(x_2) - f(x_1)}\right) = b$$

and hence $\exists \bar{y} \in \mathbb{R}^n$ such that $Q\bar{y} = b$. It follows also that $Qu_1 = Qx_1 + q - f(x_1)Q\bar{y}$ which implies $q = Q(u_1 - x_1 + f(x_1)\bar{y})$ and hence $\exists \bar{x} \in \mathbb{R}^n$ such that $Q\bar{x} = q$.

Suppose now that $\exists \bar{x}, \bar{y} \in \mathbb{R}^n$ such that $Q\bar{x} = q$ and $Q\bar{y} = b$; then

$$\nabla f(x) = Q\left(\frac{x + \bar{x} - f(x)\bar{y}}{b^T x + b_0}\right), \tag{6}$$

so that $\nabla f(x) \in Q(\mathbb{R}^n)$ $\forall x \in X$.

(ii) From (6) we have that $Qu = \nabla f(x)$ if and only if

$$Q\left(u - \frac{x + \bar{x} - f(x)\bar{y}}{b^T x + b_0}\right) = 0$$

and this happens if and only if

$$\left(u - \frac{x + \bar{x} - f(x)\bar{y}}{b^T x + b_0}\right) = k \in \ker(Q).$$

(iii) Since $Qx = Q\bar{x} = q$, it is $Q(x - \bar{x}) = 0$ and hence $x = \bar{x} + k$ with $k \in \ker(Q)$. The result then follows being $b^T k = \bar{y}^T Qk = 0 = \bar{x}^T Qk = q^T k$ $\forall k \in \ker(Q)$.

(iv) Analogous to (iii).

(v) Just note that:

$$\begin{aligned} u^T Qu &= \frac{1}{(b^T x + b_0)^2}\left[(x + \bar{x} - f(x)\bar{y})^T Q(x + \bar{x} - f(x)\bar{y})\right] \\ &= \frac{1}{(b^T x + b_0)^2}\left[(x + \bar{x} - f(x)\bar{y})^T (Qx + q - f(x)b)\right] \\ &= \frac{(f(x))^2 b^T \bar{y} + 2f(x)(b_0 - b^T \bar{x}) + (q^T \bar{x} - 2q_0)}{(b^T x + b_0)^2}. \end{aligned}$$ □

Using Lemma 8 we are able to state the following result, related to Condition (iv-a) in Theorem 7. This new lemma will be a key tool in characterizing the generalized linearity of f.

LEMMA 9. *Let us consider function f defined in* (3). *It results*

$$\nu_0(Q)=n-1, \nabla f(x)\in Q(\mathbb{R}^n)\ \forall x\in X,$$

if and only if there exist $\alpha,\beta,\gamma\in\mathbb{R},\ \alpha\neq 0$, *such that*

$$f(x)=\alpha b^T x+\beta+\frac{\alpha\gamma}{b^T x+b_0}.$$

Moreover it results $\nabla f(x)\neq 0\ \forall x\in X$ *if and only if* $\gamma\leqslant 0$.

Proof. $\Leftarrow$) By means of simple calculations $Q=[2\alpha bb^T]$ and hence $v_0(Q)=n-1$. Since $\nabla f(x)=\alpha\left[1-\gamma/(b^T x+b_0)^2\right]b$, it is $\nabla f(x)\in Q(\mathbb{R}^n), \forall x\in X$.
$\Rightarrow$) It follows from Lemma 8 that our assumption becomes

$$\nu_0(Q)=n-1 \text{ and } \exists \bar{x},\bar{y}\in\mathbb{R}^n \text{ such that } Q\bar{x}=q \text{ and } Q\bar{y}=b.$$

Since $b\neq 0$ and $\dim(Q(\mathbb{R}^n))=1$, it is $Q\bar{x}=q$ if and only if $\exists\mu\in\mathbb{R}$ such that $q=\mu b$.
Since $b\in Q(\mathbb{R}^n)$ and $\dim(Q(\mathbb{R}^n))=1$, vector b is eigenvector of Q. Therefore, being Q symmetric, there exists $\alpha\in\mathbb{R},\alpha\neq 0$ such that $Q=[2\alpha bb^T]$ and $\bar{y}=\frac{1}{2\alpha\|b\|^2}b,\ 2b^T\bar{y}=\frac{1}{\alpha}$. Consequently

$$\begin{aligned} f(x)&=\frac{\alpha(b^T x)^2+\mu b^T x+q_0}{b^T x+b_0}=\\ &=\frac{\alpha[(b^T x+b_0)-b_0]^2+\mu b^T x+\mu b_0-\mu b_0+q_0}{b^T x+b_0}\\ &=\alpha b^T x+(\mu-\alpha b_0)+\frac{\alpha b_0^2-\mu b_0+q_0}{b^T x+b_0}. \end{aligned}$$

The result then follows defining $\beta=(\mu-\alpha b_0)$ and $\gamma=b_0^2+\frac{1}{\alpha}(q_0-\mu b_0)$.

To prove the second part of the lemma, note that

$$\nabla f(x)=\alpha\left[1-\frac{\gamma}{(b^T x+b_0)^2}\right]b$$

with $\alpha\neq 0,\ b\neq 0$. Hence it results $\nabla f(x)\neq 0\ \forall x\in X$ if and only if

$$\left(b^T x+b_0\right)^2\neq\gamma\ \forall x\in X. \tag{7}$$

By definition $\{y\in\mathbb{R}: y=b^T x+b_0,\ x\in X\}=\mathbb{R}_{++}$ so that (7) holds if and only if $\gamma\leqslant 0$. □

We are now ready to provide the following characterization of quadratic fractional generalized linear functions.

THEOREM 10. *Function f defined in (3) is pseudolinear (or quasilinear) on X if and only if f is affine or there exist $\alpha,\beta,\gamma\in\mathbb{R}$, $\alpha\neq 0$, such that f can be rewritten in the following form:*

$$f(x)=\alpha b^T x+\beta+\frac{\alpha\gamma}{b^T x+b_0} \text{ with } \gamma<0.$$

Proof. $\Rightarrow$) Since f is pseudolinear, either condition (iv-a) or condition (iv-b) of Theorem 7 holds. If (iv-a) is satisfied the results follows from Lemma 9 noticing that f is affine when $\gamma=0$. Suppose now that condition (iv-b) holds. It follows from Lemma 8 that

$$\nu_-(Q)=\nu_+(Q)=1,$$
$$\exists\bar{x},\bar{y}\in\mathbb{R}^n \text{ s.t. } Q\bar{x}=q \text{ and } Q\bar{y}=b,\ p(x)=0\ \forall x\in X$$

with $\nabla f(x)\neq 0\ \forall x\in X$. Since f is not a constant function, it is $p(x)=0\ \forall x\in X$ if and only if $b^T\bar{y}=0$, $b^T\bar{x}=b_0$ and $q^T\bar{x}=2q_0$.

Since $\nu_-(Q)=\nu_+(Q)=1$, from the canonical form of Q we get $Q=[uu^T-vv^T]$ where u and v are eigenvectors of Q with $u^Tv=0$. From $Q\bar{y}=b$, $b^T\bar{y}=\bar{y}^TQ\bar{y}=0$ we have

$$\bar{y}^TQ\bar{y}=\left(u^T\bar{y}\right)^2-(v^T\bar{y})^2=0$$

so that $v^T\bar{y}=\pm\, u^T\bar{y}$. Then

$$b=Q\bar{y}=u\left(u^T\bar{y}\right)-\left(v^T\bar{y}\right)v=\left(u^T\bar{y}\right)(u+\delta v),$$

where $\delta=\pm 1$. By defining $a=\frac{1}{2u^T\bar{y}}(u-\delta v)$ and performing simple calculations we get

$$\left(ab^T+ba^T\right)=[uu^T-vv^T]=Q$$

Note that a and b are linearly independent. Let $\bar{x}\in\mathbb{R}^n$ such that $Q\bar{x}=q$ and define $a_0=a^T\bar{x}$. It results

$$q = ab^T\bar{x}+ba^T\bar{x}=ab_0+ba_0$$
$$q_0 = \frac{1}{2}q^T\bar{x}=\frac{1}{2}b_0a^T\bar{x}+a_0b^T\bar{x}=a_0b_0$$
$$\frac{1}{2}x^TQx+q^Tx+q_0 = (b^Tx+b_0)(a^Tx+a_0),$$

hence $f(x)=a^Tx+a_0$.

$\Leftarrow$) If f is affine it is trivially pseudolinear. The whole result then follows directly from Lemma 9 and Theorem 7. □

REMARK 11. The proof of Theorem 10 points out that:

(i) when $\nu_-(Q)=\nu_+(Q)=1$ f is generalized linear if and only if it is affine,
(ii) f may be affine when $\nu_0(Q)=n-1$ (case $\gamma=0$),
(iii) f is generalized linear but not affine only if $\nu_0(Q)=n-1$ (case $\gamma<0$).

Referring to Theorem 10, when $\gamma>0$ it is not possible to have a generalized linear function on the whole set X (see Example 13). However, in this case, it is possible to prove that the function may be generalized linear at least on two disjoint convex sets.

COROLLARY 12. *Consider function f defined in* (3) *and suppose that there exist $\alpha,\beta,\gamma\in\mathbb{R}$, $\alpha\neq 0$, such that f can be rewritten in the following form*:

$$f(x)=\alpha b^T x+\beta+\frac{\alpha\gamma}{b^T x+b_0}$$

(i) *if $\gamma\leqslant 0$ then f is pseudolinear* (*quasilinear*) *on X.*
(ii) *if $\gamma>0$ then f is pseudolinear* (*quasilinear*) *on*
$X_1=\{x\in\mathbb{R}^n: b^T x+b_0>\sqrt{\gamma}\}$ *and* $X_2=\{x\in\mathbb{R}^n: 0<b^T x+b_0<\sqrt{\gamma}\}$.

Proof. (i) It has already been proved in Theorem 10.
(ii) Observe that $\nabla f(x) = \frac{\alpha}{(b^T x + b_0)^2}[(b^T x + b_0)^2 - \gamma]b$, and consequently $\nabla f(x)\neq 0$ on X_1 and X_2. The result trivially follows from Theorem 7, being $Q=[2\alpha bb^T]$. □

The following examples use conditions in Theorems 7 and 10 in order to check the generalized linearity of three quadratic fractional functions.

EXAMPLE 13. Consider problem (3) where

$$f(x)=\frac{9x_1^2+24x_1x_2+16x_2^2+6x_1-8x_2+1}{3x_1+4x_2}.$$

Observe that f is not pseudolinear since it is not constant and $\nabla f(x)$ vanishes at $3x_1+4x_2=1$. In this case we get:

$$Q=\begin{bmatrix}18 & 24\\ 24 & 32\end{bmatrix}, q=\begin{bmatrix}6\\ 8\end{bmatrix}, b=\begin{bmatrix}3\\ 4\end{bmatrix}, b_0=0,\ q_0=1;$$

by simple calculations we obtain $\nu_0(Q)=1=\nu_+(Q)$ and $f(x)=3x_1+4x_2+2+\frac{1}{3x_1+4x_2}$ hence $\gamma=1>0$.

EXAMPLE 14. Consider problem (3) where

$$f(x)=\frac{8x_1^2+2x_2^2+18x_3^2-8x_1x_2-24x_1x_3+12x_2x_3+10x_1-5x_2-15x_3-4}{-2x_1+x_2+3x_3-3}.$$

In this case we get:

$$Q=\begin{bmatrix} 16 & -8 & -24 \\ -8 & 4 & 12 \\ -24 & 12 & 36 \end{bmatrix}, q=\begin{bmatrix} 10 \\ -5 \\ -15 \end{bmatrix}, b=\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, b_0=-3,\ q_0=-4$$

Since Q is positive semidefinite with $\nu_0(Q)=n-1$, we have to verify condition iv-a) of Theorem 7. Note that $\nabla f(x)=\left(2+\frac{1}{(-2x_1+x_2+3x_3-3)^2}\right)b\neq 0\ \forall x\in X,\ \nabla f(x)\in Q(\mathbb{R}^2)$, hence f is pseudolinear. The same result can be obtained by means of Theorem 10. In fact simple calculations give

$$f(x)=-4x_1+2x_2+6x_3+1-\frac{1}{-2x_1+x_2+3x_3-3},$$

so that $\alpha=2,\ \gamma=-\dfrac{1}{2}<0$ and hence f is pseudolinear.

EXAMPLE 15. Consider problem (3) where

$$f(x)=\frac{-8x_1^2-24x_2^2+32x_3^2+16x_1x_2+64x_2x_3+16x_2+16x_3+2}{-8x_1-8x_2-16x_3-4}.$$

In this case we get:

$$Q=\begin{bmatrix} -16 & 16 & 0 \\ 16 & 48 & 64 \\ 0 & 64 & 64 \end{bmatrix}, q=\begin{bmatrix} 0 \\ 16 \\ 16 \end{bmatrix}, b=\begin{bmatrix} -8 \\ -8 \\ -16 \end{bmatrix}, b_0=-4,\ q_0=2$$

Q is indefinite with $\nu_+(Q)=\nu_-(Q)=1$, hence f is pseudolinear if and only if it is affine (see also Remark 11). In fact, by means of simple calculations we obtain

$$f(x)=x_1-3x_2-2x_3-\frac{1}{2},$$

4. A Larger Class of Quadratic Fractional Functions

The aim of this section is to study a class of functions larger than the one considered so far. Specifically speaking, we aim to characterize the generalized linearity of the following type of functions:

$$g(x)=\frac{\frac{1}{2}x^TQx+q^Tx+q_0}{b^Tx+b_0}+c^Tx=f(x)+c^Tx \tag{8}$$

where as usual $X=\{x\in\mathbb{R}^n : b^Tx+b_0>0\}$, $n\geqslant 2$, Q is a $n\times n$ symmetric matrix, $q,x,b,c\in\mathbb{R}^n$, $b\neq 0$, and $q_0,b_0\in\mathbb{R}$. Note that g is of the kind (3) when $c=0$ and $Q\neq 0$. First of all observe that it may happen that f $[g]$ is pseudolinear even if g $[f]$ is not. This is pointed out in the following examples.

EXAMPLE 16. Consider problem (8) where

$$g(x)=\frac{x_1^2+x_1x_2-x_1+2x_2+1}{x_1+x_2}-x_1.$$

Observe that

$$f(x)=\frac{x_1^2+x_1x_2-x_1+2x_2+1}{x_1+x_2}=x_1-1+\frac{3x_2+1}{x_1+x_2}$$

is not pseudolinear while

$$g(x)=\frac{-x_1+2x_2+1}{x_1+x_2}$$

is pseudolinear since it is a linear fractional function.
Consider now problem (8) where

$$g(x)=\frac{8x_1^2+8x_1x_2+2x_2^2-1}{2x_1+x_2}+x_1-x_2.$$

Observe that

$$f(x)=\frac{8x_1^2+8x_1x_2+2x_2^2-1}{2x_1+x_2}=2(2x_1+x_2)-\frac{1}{2x_1+x_2}$$

is pseudolinear while

$$g(x)=5x_1+x_2-\frac{1}{2x_1+x_2}$$

is not pseudolinear.

The characterization of the generalized linearity of g follows from Theorem 10.

THEOREM 17. *Let g be of the kind* (8)*; the following statements hold:*

(i) *g is affine if and only if f is affine;*

(ii) *g is pseudolinear* (*quasilinear*) *but not affine if and only if either $Q+bc^T+cb^T=0$ or there exist $\alpha,\xi\in\mathbb{R}$, $\alpha\neq 0$, such that*:

$$Q+bc^T+cb^T=2\alpha bb^T,\quad q+b_0c=\xi b,\quad b_0^2<\frac{\xi b_0-q_0}{\alpha}$$

Proof. (i) The result is trivial provided that g is the sum of f and an affine function.

(ii) By means of simple calculations g can be rewritten as follows

$$g(x)=\frac{\frac{1}{2}x^T[Q+bc^T+cb^T]x+[q+b_0c]^Tx+q_0}{b^Tx+b_0}.$$

If $Q+bc^T+cb^T=[0]$ then g is a linear fractional function which is known to be pseudolinear. If otherwise $Q+bc^T+cb^T\neq[0]$, from Theorem 10 g is pseudolinear but not affine if and only if there exist $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$, such that it can be rewritten in the following form:

$$g(x) = \alpha b^T x + \beta + \frac{\alpha\gamma}{b^T x + b_0} \text{ with } \gamma < 0$$

and so

$$g(x) = \frac{\frac{1}{2} x^T [2\alpha b b^T] x + (\beta + \alpha b_0) b^T x + (\beta b_0 + \alpha\gamma)}{b^T x + b_0}.$$

This means that:

$$Q + bc^T + cb^T = 2\alpha b b^T, \quad q + b_0 c = (\beta + \alpha b_0) b, \quad q_0 = \beta b_0 + \alpha\gamma.$$

Defining $\xi = \beta + \alpha b_0$, so that $\beta = \xi - \alpha b_0$ and $\gamma = \dfrac{q_0 - \beta b_0}{\alpha} = b_0^2 - \dfrac{\xi b_0 - q_0}{\alpha}$, the result then follows from $\gamma < 0$. □

Theorem 17 can be applied also to functions of the kind (3) taking $c = 0$. The next examples clarify the use of the conditions given in Theorem 17.

EXAMPLE 18. Consider again function g in Example 16. Observe that

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \ q = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ q_0 = 1, \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b_0 = 1, \ c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$Q + bc^T + cb^T = 0$ and hence g is pseudolinear.

EXAMPLE 19. Consider problem (8) where

$$g(x) = \frac{x_2^2 - 2x_1x_2 + 3x_2x_3 - 2x_1 - 2x_2 + 3x_3 - 4}{-2x_1 + x_2 + 3x_3 - 3} - 4x_1 + x_2 + 6x_3.$$

Observe that since

$$Q = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}, \ q = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}, \ b = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \ c = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix},$$

$q_0 = -4$, $b_0 = -3$, it results

$$Q + bc^T + cb^T = \begin{bmatrix} 16 & -8 & -24 \\ -8 & 2 & 12 \\ -24 & 12 & 36 \end{bmatrix} = 2\alpha b b^T \text{ with } \alpha = 2,$$

$$q + b_0 c = \begin{bmatrix} 10 \\ -5 \\ -15 \end{bmatrix} = \xi b \text{ with } \xi = -5,$$

$$b_0^2 = 9 < \frac{19}{2} = \frac{\xi b_0 - q_0}{\alpha},$$

hence g is pseudolinear.

5. Pseudolinear Quadratic Fractional Functions in Optimization

Theorem 10 shows that every generalized linear quadratic fractional function f is the sum of a linear and a linear fractional one. This property, together with Theorem 17 and Corollary 12, can be efficiently used in order to study the following class of problems

$$\min_{x\in S\subseteq X}/\max \quad g(x)=\frac{\frac{1}{2}x^TQx+q^Tx+q_0}{b^Tx+b_0}+c^Tx=f(x)+c^Tx, \tag{9}$$

where g is of the kind (8) and the matrix $Q+bc^T+cb^T$ has at least $n-2$ zero eigenvalues. Note that recent optimality conditions for pseudolinear functions can be found in [17, 24].

Case 1 – g is an affine function

For Theorem 17, this case occurs whenever f is an affine function that is when:

$\exists a_0 \in \mathbb{R}$ such that $q_0 = a_0b_0$, $\exists a \in \mathbb{R}^n$ such that $q-a_0b = b_0a$, and it results $Q=ab^T+ba^T$.

In this case $f(x) = a^Tx + a_0$ and hence $g(x) = (a + c)^Tx + a_0$. Problem (9) can then be solved by means of a linear problem:

$$\operatorname*{argmin/argmax}_{x\in S\subseteq X} \; g(x)=\operatorname*{argmin/argmax}_{x\in S\subseteq X} \; (a+c)^Tx$$

Case 2 – g is a linear fractional function

This case occurs when (see Theorem 17):

$$Q+bc^T+cb^T=0$$

The objective function becomes

$$g(x)=\frac{(q+b_0c)^Tx+q_0}{b^Tx+b_0}$$

and problem (9) can be solved with any algorithm for linear fractional functions (see for all [8, 9, 16]).

Case 3 – g is a pseudolinear but not affine function

Suppose that g is not a linear fractional function (this case has been already considered). By means of Theorem 17 g is pseudolinear but not affine when:

$\exists\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $Q + bc^T + cb^T = 2\alpha bb^T$, $\exists\xi \in \mathbb{R}$ such that $q+b_0c=\xi b$, and it results $b_0^2<(\xi b_0-q_0)/\alpha$.

Defining $\beta = \xi - \alpha b_0$, $\gamma = b_0^2 - [(\xi b_0 - q_0)/\alpha]$ and the function $\varphi(t) = \alpha t + \beta + [\alpha\gamma/(t + b_0)]$ we have that:

$$g(x) = \alpha b^T x + \beta + \frac{\alpha\gamma}{b^T x + b_0} = \varphi(b^T x) \quad \text{with} \quad \gamma < 0.$$

Since $\varphi'(t) = \alpha\left(1 - \gamma/(t + b_0)^2\right)$ we have that $\varphi'(t) > 0$ $[<0]$ if and only if $\alpha > 0$ $[<0]$ and hence:

$$\alpha > 0 \Rightarrow \operatorname*{argmin/argmax}_{x \in S \subseteq X} g(x) = \operatorname*{argmin/argmax}_{x \in S \subseteq X} b^T x,$$

$$\alpha > 0 \Rightarrow \operatorname*{argmin/argmax}_{x \in S \subseteq X} g(x) = \operatorname*{argmax/argmin}_{x \in S \subseteq X} b^T x.$$

Consequently problem (9) can be solved by means of a linear one.

Case 4 – g is pseudolinear on subsets of X

Suppose that:

$\exists \alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $Q + bc^T + cb^T = 2\alpha bb^T$, $\exists \xi \in \mathbb{R}$ such that $q + b_0 c = \xi b$, and it results $b_0^2 > (\xi b_0 - q_0)/\alpha$.

Defining $\beta = \xi - \alpha b_0$, $\gamma = b_0^2 - [(\xi b_0 - q_0)/\alpha]$ and the function $\varphi(t) = \alpha t + \beta + [\alpha\gamma/(t + b_0)]$, from Theorem 17, we have that:

$$g(x) = \alpha b^T x + \beta + \frac{\alpha\gamma}{b^T x + b_0} = \varphi(b^T x) \quad \text{with} \quad \gamma > 0.$$

From Corollary 12 g is not pseudolinear on X but it is pseudolinear on

$$X_1 = \{x \in \mathbb{R}^n : b^T x + b_0 > \sqrt{\gamma}\} \text{ and}$$
$$X_2 = \{x \in \mathbb{R}^n : 0 < b^T x + b_0 < \sqrt{\gamma}\}.$$

Assume now $X_3 = \{x \in \mathbb{R}^n : b^T x + b_0 = \sqrt{\gamma}\}$. Since

$$g(x) = 2\alpha\sqrt{\gamma} - \alpha b_0 + \beta = 2\alpha(\sqrt{\gamma} - b_0) + \xi \quad \forall x \in X_3,$$

g is constant on X_3. Consequently, problem (9) can be studied by determining the sets

$$S_1^m = \operatorname*{argmin}_{x \in S \cap X_1}\{b^T x\},\ S_1^M = \operatorname*{argmax}_{x \in S \cap X_1}\{b^T x\},$$
$$S_2^m = \operatorname*{argmin}_{x \in S \cap X_2}\{b^T x\},\ S_2^M = \operatorname*{argmax}_{x \in S \cap X_2}\{b^T x\},$$

and by denoting (if the related sets are nonempty)

$$x_1^m \in S_1^m, \quad x_1^M \in S_1^M, \quad x_2^m \in S_2^m, \quad x_2^M \in S_2^M, \quad x_3 \in S \cap X_3.$$

If $\alpha > 0$ then $\varphi'(t) > 0$ when $t + b_0 > \sqrt{\gamma}$ while it is $\varphi'(t) < 0$ when $0 < t + b_0 < \sqrt{\gamma}$. Hence we get

$$\min_{x \in S \subseteq X} \{g(x)\} = \min\{g(x_1^m), g(x_2^M), g(x_3)\},$$

$$\max_{x \in S \subseteq X} \{g(x)\} = \max\{g(x_1^M), g(x_2^m), g(x_3)\};$$

analogously for $\alpha < 0$ it is

$$\min_{x \in S \subseteq X} \{g(x)\} = \min\{g(x_1^M), g(x_2^m), g(x_3)\},$$

$$\max_{x \in S \subseteq X} \{g(x)\} = \max\{g(x_1^m), g(x_2^M), g(x_3)\}.$$

Again, problem (9) can be solved by means of linear ones.

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